

Derivada de Funciones de una Variable

Efraín Martínez M.

Diciembre 2022

Resumen

La derivada, uno de los temas más importantes del cálculo diferencial e integral, probablemente con mayor aplicación en diferentes ramas de la ingeniería, economía, etc.

1. Reglas de derivación

Sea, $u = u(x)$, $v = v(x)$, $w = w(x)$ funciones de x ; a, b, c, n, p, q constantes, entonces:

- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(cx) = c$
- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(x^{p/q}) = \frac{p}{q}x^{p/q-1}$
- $\frac{d}{dx}|x| = \frac{x}{|x|} = \begin{cases} 1 & x > 0; \\ -1 & x < 0. \end{cases}$
- $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$
- $\frac{d}{dx}(u \pm v \pm w) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx}$
- $\frac{d}{dx} \sum_{i=1}^n u_i = \sum_{i=1}^n \frac{du_i}{dx}$
- $\frac{d^n}{dx^n}(u \cdot v) = y^{(n)} = u^{(n)} + nu^{(n-1)}v' + \frac{n(n-1)}{1 \cdot 2}u^{(n-2)}v'' + \dots + uv^{(n)}$, fórmula de *Leibniz*.
- $\frac{d}{dx}(g \circ f)(x) = \frac{d}{dx}g(f(x)) = g'(f(x)) \cdot f'(x)$, derivada de g compuesta con f
- $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ $y = g(u)$, $u = f(x)$, regla de la cadena
- $\frac{d}{dx}(cu) = c \frac{du}{dx}$
- $\frac{d}{dx}(u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx}$
- $\frac{d}{dx}(u \cdot v \cdot w) = vw \frac{du}{dx} + uw \frac{dv}{dx} + uv \frac{dw}{dx}$
- $\frac{d}{dx} \prod_{i=1}^n u_i = \sum_{i=1}^n \left[\prod_{j=1, j \neq i}^n (u_j) \frac{du_i}{dx} \right]$
- $\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$
- $\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$
- $\frac{d}{dx}(u^{p/q}) = \frac{p}{q}u^{p/q-1} \frac{du}{dx}$
- $\frac{d}{dx}|u| = \frac{u}{|u|} \cdot u' = \begin{cases} du/dx & u > 0; \\ -du/dx & u < 0. \end{cases}$

21. $\frac{d}{dx}(h \circ g \circ f)(x) = h'(g(f(x))) \cdot g'(f(x)) \cdot f'(x)$, derivada de $(h \circ g \circ f)$

22. $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$ $y = h(v)$, $v = g(u)$, $u = f(x)$, regla de la cadena

23. $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$, $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, derivada de función inversa

24. $\frac{d}{dx} \sum_{i=0}^n a_i x^i = \sum_{i=1}^n i a_i x^{i-1}$ 25. $\frac{d}{dx} \sqrt[n]{x} = \frac{1}{n \sqrt[n]{x^{n-1}}}$

26. $\frac{dA(x)}{dx} = \begin{pmatrix} f_{11}(x) & \cdots & f_{1n}(x) \\ \vdots & \ddots & \vdots \\ f_{i1}(x) & \cdots & f_{in}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mn}(x) \end{pmatrix}' = \begin{pmatrix} f'_{11}(x) & \cdots & f'_{1n}(x) \\ \vdots & \ddots & \vdots \\ f'_{i1}(x) & \cdots & f'_{in}(x) \\ \vdots & \ddots & \vdots \\ f'_{m1}(x) & \cdots & f'_{mn}(x) \end{pmatrix}$, donde: $A(x) = (f_{ij}(x))_{(mn)}$

27. $\frac{d|A(x)|}{dx} = \begin{vmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{vmatrix}' = \begin{vmatrix} f'_{11}(x) & f'_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{vmatrix} + \begin{vmatrix} f_{11}(x) & f_{12}(x) \\ f'_{21}(x) & f'_{22}(x) \end{vmatrix}$

28. $\frac{d|A(x)|}{dx} = \begin{vmatrix} f_{11}(x) & \cdots & f_{1n}(x) \\ \vdots & \ddots & \vdots \\ f_{i1}(x) & \cdots & f_{in}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mn}(x) \end{vmatrix}' = \sum_{i=1}^n \begin{vmatrix} f_{11}(x) & \cdots & f_{1n}(x) \\ \vdots & \ddots & \vdots \\ f'_{i1}(x) & \cdots & f'_{in}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mn}(x) \end{vmatrix}$, donde: $|A(x)| = |(f_{ij}(x))_{(mn)}|$

2. Derivada de funciones exponenciales y logarítmicas

1. $(a^x)' = a^x \ln a$, $0 < a \neq 1$

3. $(x^x)' = x^x (\ln x + 1)$

2. $(e^x)' = e^x$

4. $(u^v)' = u^v \left(\ln u \frac{dv}{dx} + \frac{v}{u} \frac{du}{dx} \right)$

1. $(\ln x)' = \frac{1}{x}$, $x > 0$

3. $(\log_a x)' = \frac{\log_a e}{x}$ $0 < a \neq 1$

2. $(\log x)' = \frac{\log e}{x}$ $x > 0$

4. $(\log_v u)' = \frac{v \ln v \cdot \frac{du}{dx} - u \ln u \cdot \frac{dv}{dx}}{u \cdot v \ln^2 v}$

3. Derivada de funciones trigonométricas

1. $(\sin x)' = \cos x$

3. $(\tan x)' = \sec^2 x$

5. $(\sec x)' = \sec x \tan x$

2. $(\cos x)' = -\sin x$

4. $(\cot x)' = -\csc^2 x$

6. $(\csc x)' = -\csc x \cot x$

1. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

3. $(\arctan x)' = \frac{1}{1+x^2}$

5. $(\operatorname{arcsec} x)' = \frac{1}{x\sqrt{x^2-1}}$

2. $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

4. $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$

6. $(\operatorname{arccsc} x)' = -\frac{1}{x\sqrt{x^2-1}}$

4. Derivada de funciones hiperbólicas

- | | |
|-----------------------------------------------------------|----------------------------------------------------------------|
| 1. $(\sinh x)' = \cosh x$ | 4. $(\coth x)' = -\operatorname{csch}^2 x$ |
| 2. $(\cosh x)' = \sinh x$ | 5. $(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$ |
| 3. $(\tanh x)' = \operatorname{sech}^2 x$ | 6. $(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$ |
| 1. $(\operatorname{argsinh} x)' = \frac{1}{\sqrt{1+x^2}}$ | 4. $(\operatorname{argcoth} x)' = -\frac{1}{x^2-1}$ |
| 2. $(\operatorname{argcosh} x)' = \frac{1}{\sqrt{x^2-1}}$ | 5. $(\operatorname{argsech} x)' = -\frac{1}{x\sqrt{1-x^2}}$ |
| 3. $(\operatorname{argtanh} x)' = \frac{1}{1-x^2}$ | 6. $(\operatorname{argcsch} x)' = -\frac{1}{x\sqrt{x^2+1}}$ |

5. Derivada de funciones paramétricas

Si una función $y = f(x)$ está definida en forma paramétrica:

$$\begin{cases} x=f(t) \\ y=g(t) \end{cases} \quad \alpha < t < \beta$$

donde: $x = f(t)$, $y = g(t)$ son funciones derivables, tal que $f'(t) \neq 0$ y $x = f(t)$ es inversible $t = f^{-1}(x)$, entonces:

$$(1) \quad \frac{dy}{dx} = \frac{y'_t}{x'_t} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$(2) \quad \frac{d^2y}{dx^2} = \frac{x'_t y''_t - y'_t x''_t}{(x'_t)^3} = \frac{\begin{vmatrix} x'_t & y'_t \\ x''_t & y''_t \end{vmatrix}}{(x'_t)^3}$$

6. Derivada de funciones implícitas

Si, la ecuación $F(x, y) = 0$ define a y como función implícita de x , $y = y(x)$ derivable en x , entonces:

$$(3) \quad \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}, \quad \frac{\partial F}{\partial y} \neq 0$$

$$(4) \quad \frac{d^2y}{dx^2} = -\frac{\frac{\partial^2 F}{\partial x^2} \left(\frac{\partial F}{\partial y}\right)^2 - 2\frac{\partial^2 F}{\partial x y} \cdot \frac{\partial F}{\partial x} \cdot \frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial y^2} \left(\frac{\partial F}{\partial x}\right)^2}{\left(\frac{\partial F}{\partial y}\right)^3}, \quad \frac{\partial F}{\partial y} \neq 0$$

$$(5) \quad \frac{d^2y}{dx^2} = \frac{1}{\left(\frac{\partial F}{\partial y}\right)^3} \begin{vmatrix} 0 & \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial x} & \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x y} \\ \frac{\partial F}{\partial y} & \frac{\partial^2 F}{\partial y x} & \frac{\partial^2 F}{\partial y^2} \end{vmatrix}, \quad \frac{\partial F}{\partial y} \neq 0$$

Mayores detalles de estas fórmulas se pueden encontrar en el libro de Cálculo Diferencial e Integral de Funciones de una Variable del autor.

Cualquier error es responsabilidad del autor¹, sugerencias a la dirección que aparece en pie de página, gracias.

¹Email: eframath@hotmail.com; SitioWeb: <https://www.eframath.com>