Derivada de Funciones de una Variable Real

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Resumen

La derivada, uno de lo temas más importantes del cálculo diferencial e integral, probablemente con mayor aplicación en diferentes ramas de la ingeniería, economía, etc.

0.1. Reglas de derivación

Sea, $u = u(x), v = v(x), w = (x)$ funciones de x; a, b, c, n, p, q constantes, entonces:

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(x) = 1$
3. $\frac{d}{dx}(cx) = c$
4. $\frac{d}{dx}(x^n) = nx^{n-1}$
5. $\frac{d}{dx}\left(x^{p/q}\right) = \frac{p}{q}x^{p/q-1}$
6. $\frac{d}{dx}|x| = \frac{x}{|x|} = \begin{cases} 1 & \text{x ¿0;} \\ -1 & \text{x ¡0.} \end{cases}$
7. $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$
8. $\frac{d}{dx}(u \pm v \pm w) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx}$
9. $\frac{d}{dx}\sum_{i=1}^{n} u_i = \sum_{i=1}^{n}\frac{du_i}{dx}$
10. $\frac{d}{dx}(cu) = c\frac{du}{dx}$
11. $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$
12. $\frac{d}{dx}(u \cdot v \cdot w) = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}$
13. $\frac{d}{dx}\prod_{i=1}^{n} u_i = \sum_{i=1}^{n}\left[\prod_{j=1, j\neq i}^{n}(u_j)\frac{du_i}{dx}\right]$
14. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,(du/dx) - u\,(dv/dx)}{v^2}$
15. $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$
16. $\frac{d}{dx}\left(u^{p/q}\right) = \frac{p}{q}u^{p/q-1}\frac{du}{dx}$
17. $\frac{d}{dx}|\,u| = \frac{u}{|\,u|} \cdot u' = \begin{cases} du/dx & \text{u ¿0;} \\ -du/dx & \text{u ¡0.} \end{cases}$
18. $\frac{d^n}{dx^n}(u \cdot v) = y^{(n)} = u^{(n)} + nu^{n-1}v' + \frac{n(n-1)}{1 \cdot 2}u^{n-2}v'' + \cdots + uv^n,$ fórmula de *Leibniz.*
19. $\frac{d}{dx}\big(g \circ f\big)(x) = \frac{d}{dx}g\big(f(x)\big) = g'\big(f(x)\big) \cdot f'(x),$ derivada de g compuesta con f
20. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ $\quad y = g(u),\ u = f(x),$ regla de la cadena
21. $\frac{d}{dx}(h \circ g \circ f)(x) = h'\big(g\big(f(x)\big)\big) \cdot g'\big(f(x)\big) \cdot f'(x),$ derivada de $(h \circ g \circ f)$

22. $\dfrac{dy}{dx} = \dfrac{dy}{dv} \cdot \dfrac{dv}{du} \cdot \dfrac{du}{dx}$ $\qquad y = h(v),\ v = g(u),\ u = f(x),$ $\qquad$ regla de la cadena

23. $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}, \qquad \dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}},$ $\qquad$ derivada de función inversa

24. $\dfrac{d}{dx} \displaystyle\sum_{i=0}^{n} a_i x^i = \sum_{i=1}^{n} i a_i x^{i-1}$

25. $\dfrac{d}{dx} \sqrt[n]{x} = \dfrac{1}{n \sqrt[n]{x^{n-1}}}$

26. $\dfrac{d\,A(x)}{dx} = \begin{pmatrix} f_{11}(x) & \cdots & f_{1n}(x) \\ \cdots & \cdots & \cdots \\ f_{i1}(x) & \cdots & f_{in}(x) \\ \cdots & \cdots & \cdots \\ f_{m1}(x) & \cdots & f_{mn}(x) \end{pmatrix}' = \begin{pmatrix} f'_{11}(x) & \cdots & f'_{1n}(x) \\ \cdots & \cdots & \cdots \\ f'_{i1}(x) & \cdots & f'_{in}(x) \\ \cdots & \cdots & \cdots \\ f'_{m1}(x) & \cdots & f'_{mn}(x) \end{pmatrix}$, donde: $A(x) = \big(f_{ij}(x)\big)_{(mn)}$

27. $\dfrac{d|A(x)|}{dx} = \begin{vmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{vmatrix}' = \begin{vmatrix} f'_{11}(x) & f'_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{vmatrix} + \begin{vmatrix} f_{11}(x) & f_{12}(x) \\ f'_{21}(x) & f'_{22}(x) \end{vmatrix}$

28. $\dfrac{d|A(x)|}{dx} = \begin{vmatrix} f_{11}(x) & \cdots & f_{1n}(x) \\ \cdots & \cdots & \cdots \\ f_{i1}(x) & \cdots & f_{in}(x) \\ \cdots & \cdots & \cdots \\ f_{m1}(x) & \cdots & f_{mn}(x) \end{vmatrix}' = \displaystyle\sum_{i=1}^{n} \begin{vmatrix} f_{11}(x) & \cdots & f_{1n}(x) \\ \cdots & \cdots & \cdots \\ f'_{i1}(x) & \cdots & f'_{in}(x) \\ \cdots & \cdots & \cdots \\ f_{m1}(x) & \cdots & f_{mn}(x) \end{vmatrix}$, donde: $\big|A(x)\big| = \big|(f_{ij}(x))_{(mn)}\big|$

0.2. Derivada de funciones exponenciales y logarítmicas

1. $(a^x)' = a^x \ln a, \quad 0 < a \neq 1$
2. $(e^x)' = e^x$
3. $(x^x)' = x^x(\ln x + 1)$
4. $(u^v)' = u^v\Big(\ln u \frac{dv}{dx} + \frac{v}{u}\frac{du}{dx}\Big)$

1. $(\ln x)' = \frac{1}{x}, \quad x > 0$
2. $(\log x)' = \frac{\log e}{x} \qquad x > 0$
3. $(\log_a x)' = \frac{\log_a e}{x} \quad 0 < a \neq 1$
4. $(\log_v u)' = \frac{v \ln v \cdot \frac{du}{dx} - u \ln u \cdot \frac{dv}{dx}}{u \cdot v \ln^2 v}$

0.3. Derivada de funciones trigonométricas

1. $(\operatorname{sen} x)' = \cos x$
2. $(\cos x)' = -\operatorname{sen} x$
3. $(\tan x)' = \sec^2 x$
4. $(\cot x)' = -\csc^2 x$
5. $(\sec x)' = \sec x \tan x$
6. $(\csc x)' = -\csc x \cot x$

1. $(\operatorname{arc\,sen} x)' = \frac{1}{\sqrt{1-x^2}}$
2. $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$
3. $(\arctan x)' = \frac{1}{1+x^2}$
4. $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$
5. $(\operatorname{arcsec} x)' = \frac{1}{x\sqrt{x^2-1}}$
6. $(\operatorname{arccsc} x)' = -\frac{1}{x\sqrt{x^2-1}}$

0.4. Derivada de funciones hiperbólicas

1. $(\operatorname{senh} x)' = \cosh x$
2. $(\cosh x)' = \operatorname{senh} x$
3. $(\tanh x)' = \operatorname{sech}^2 x$
4. $(\coth x)' = -\operatorname{csch}^2 x$
5. $(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$
6. $(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$

1. $(\operatorname{argsinh} x)' = \frac{1}{\sqrt{1+x^2}}$
2. $(\operatorname{argcosh} x)' = \frac{1}{\sqrt{x^2-1}}$
3. $(\operatorname{argtanh} x)' = \frac{1}{1-x^2}$
4. $(\operatorname{argcoth} x)' = -\frac{1}{x^2-1}$
5. $(\operatorname{argsech} x)' = -\frac{1}{x\sqrt{1-x^2}}$
6. $(\operatorname{argcsch} x)' = -\frac{1}{x\sqrt{x^2+1}}$

0.5. Derivada de funciones paramétricas

Si una función $y = f(x)$ está definida en forma paramétrica:

$$\begin{cases} \text{x=f(t)} \\ \text{y=g(t)} \end{cases} \qquad \alpha < t < \beta$$

donde: $x = f(t)$, $y = g(t)$ son funciones derivables, tal que $f'(t) \neq 0$ y $x = f(t)$ es inversible $t = f^{-1}(x)$, entonces:

$$(1) \qquad \frac{dy}{dx} = \frac{y'_t}{x'_t} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$(2) \qquad \frac{d^2y}{dx^2} = \frac{x'_t y''_t - y'_t x''_t}{(x'_t)^3} = \frac{\begin{vmatrix} x'_t & y'_t \\ x''_t & y''_t \end{vmatrix}}{(x'_t)^3}$$

0.6. Derivada de funciones implícitas

Si, la ecuación $F(x, y) = 0$ define a y como función implícita de x, $y = y(x)$ derivable en x, entonces:

$$(3) \qquad \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}, \qquad \frac{\partial F}{\partial y} \neq 0$$

$$(4) \qquad \frac{d^2y}{dx^2} = -\frac{\frac{\partial^2 F}{\partial x^2}\left(\frac{\partial F}{\partial y}\right)^2 - 2\frac{\partial^2 F}{\partial xy} \cdot \frac{\partial F}{\partial x} \cdot \frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial y^2}\left(\frac{\partial F}{\partial x}\right)^2}{\left(\frac{\partial F}{\partial y}\right)^3}, \qquad \frac{\partial F}{\partial y} \neq 0$$

$$(5) \qquad \frac{d^2y}{dx^2} = \frac{1}{\left(\frac{\partial F}{\partial y}\right)^3} \begin{vmatrix} 0 & \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial x} & \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial xy} \\ \frac{\partial F}{\partial y} & \frac{\partial^2 F}{\partial yx} & \frac{\partial^2 F}{\partial y^2} \end{vmatrix}, \qquad \frac{\partial F}{\partial y} \neq 0$$

Fórmulas sujeta a demostración, las mismas que se pueden encontrar en Cálculo Diferencial e Integral de Funciones a una Variable Real.

Cualquier error es responsabilidad del autor[1], sugerencias a la dirección que aparece en pie de página, gracias.

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